Higher-order linear dynamical systems

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Outline

1 Solving higher-order systems

2 Asymptotic stability: necessary condition

3 Asymptotic stability: necessary and sufficient condition

4 Oscillations

5 Delayed feedback

The material in these slides follows:
H R Wilson (1999). Spikes, Decisions, and Actions: The Dynamical Foundations of Neuroscience. Oxford University Press.

Outline

Theorem 4: solving higher-order systems

Theorem 4: The solution to the Nth order differential equation (4.1) is obtained by finding the N roots, λ_1 to λ_N, of the characteristic equation:

$$|\overleftrightarrow{A} - \lambda \overleftrightarrow{I}| = 0$$

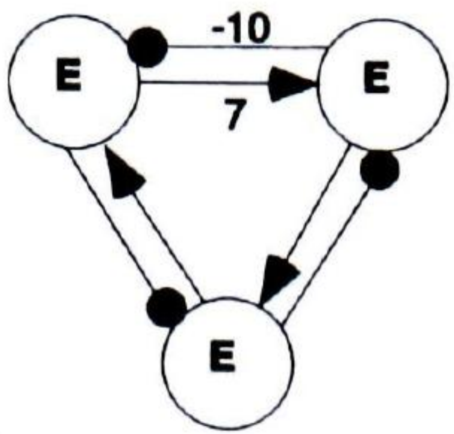
where I is the identity matrix. The components of the solution vector are then of the form:

$$\sum_{n=1}^{N} A_n \, \mathrm{e}^{\lambda_n t}$$

where values of the constants A_n are determined by solving for the eigenvectors and by the initial conditions. If a single eigenvalue, say $\lambda = \omega$, occurs k times, then the solutions associated with this eigenvalue will be of the form:

$$A_1 \, \mathrm{e}^{\omega t} + A_2 t \, \mathrm{e}^{\omega t} + A_3 t^2 \, \mathrm{e}^{\omega t} + \cdots + A_k t^{k-1} \, \mathrm{e}^{\omega t}$$

Example: a network of three nodes



$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} -5 & -10 & 7 \\ 7 & -5 & -10 \\ -10 & 7 & -5 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

Theorem 4 tells us that the states E_i will all have the form:

$$E_i = A\,\mathrm{e}^{-8t} + B\,\mathrm{e}^{-3.5t}\sin(14.7t) + C\,\mathrm{e}^{-3.5t}\cos(14.7t)$$

where -8 and $-3.5 + 14.7i$ and $-3.5 - 14.7i$ are the three eigenvalues of the system.

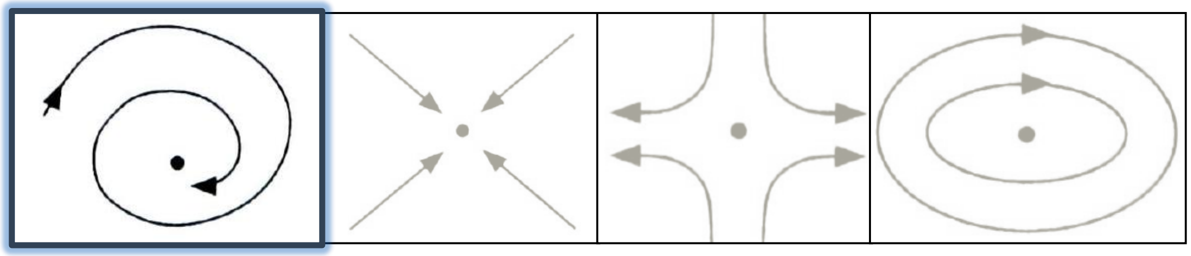
Example: solving vs. characterizing the system

Whether or not we care to solve for the constants, we can
use the eigenvalues to characterize the system:

```
eig([-5 -10 7; 7 -5 -10; -10 7 -5])
ans =
     -3.5 +     14.72243i
     -3.5 -     14.72243i
       -8
```

From Chapter 3 we know that the eqilibrium point of this system

- must be a **spiral point** (because the eigenvalues are a complex conjugate pair)
- must be **asymptotically stable** (because the real part of the eigenvalues is negative)



spiral point node saddle point centre

http://demonstrations.wolfram.com/TwoDimensionalLinearSystems/

Outline

Theorem 5: a simple necessary condition for asymptotic stability

Theorem 5: The stability, asymptotic stability, or instability of the equilibrium point of (4.1) is determined by the roots of the characteristic equation:

$$|\overleftrightarrow{A} - \lambda \overleftrightarrow{I}| = 0$$

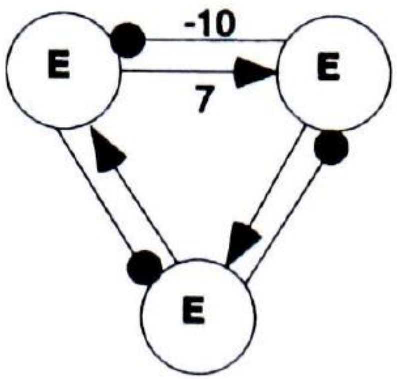
The equilibrium will be asymptotically stable if all the roots of the characteristic equation have negative real parts. Writing the characteristic equation as:

$$\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_{N-1}\lambda + a_N = 0$$

A necessary (but not sufficient) condition for all roots to have negative real parts is:

$$a_k > 0 \quad \text{for } 1 \leq k \leq N$$

Example: testing the necessary condition



$$\frac{d}{dt}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} -5 & -10 & 7 \\ 7 & -5 & -10 \\ -10 & 7 & -5 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

To check the condition from Theorem 5, we write down the characteristic equation:

$$|A - \lambda I| = 0$$

$$\left|\begin{pmatrix} -5 & -10 & 7 \\ 7 & -5 & -10 \\ -10 & 7 & -5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = 0$$

$$\lambda^3 + \underbrace{15}_{a_1>0}\lambda^2 + \underbrace{285}_{a_2>0}\lambda + \underbrace{1832}_{a_3>0} = 0$$

Thus, a necessary condition for asymptotic stability is fulfilled.

Outline

Theorem 6: a simple equivalence of asymptotic stability

Theorem 6 (Routh–Hurwitz theorem): Given the coefficients a_k of the characteristic equation in Theorem 5, compute the following series of determinants for order $N = 5$:

$$\Delta_1 = a_1, \qquad \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \qquad \Delta_4 = \begin{vmatrix} a_1 & 1 & 0 & 0 \\ a_3 & a_2 & a_1 & 1 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & a_5 & a_4 \end{vmatrix}$$

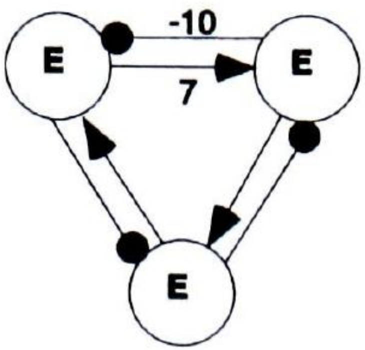
$$\Delta_5 = a_5 \Delta_4$$

The system is asymptotically stable if and only if:

$$\Delta_k > 0 \quad \text{for } 1 \le k \le N$$

Furthermore, if $\Delta_N = 0$, then $\lambda = 0$ is one eigenvalue.

Example: testing the equivalence condition



$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} -5 & -10 & 7 \\ 7 & -5 & -10 \\ -10 & 7 & -5 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

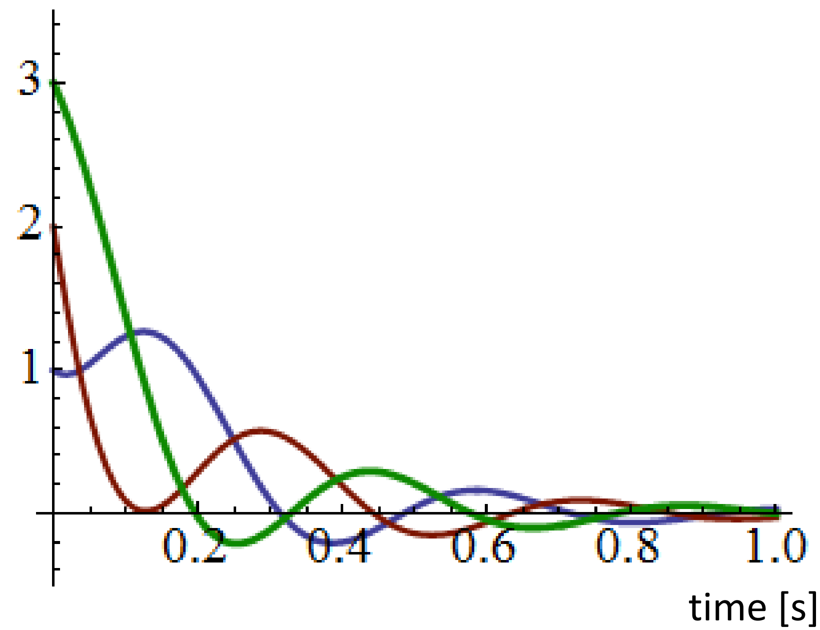
To evaluate the condition in Theorem 6, we consider three determinants:

$$\Delta_1 = a_1 = \underbrace{15}_{>0}$$

$$\Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = \begin{vmatrix} 15 & 1 \\ 1832 & 285 \end{vmatrix} = \underbrace{2443}_{>0}$$

$$\Delta_3 = a_3\Delta_2 = 1832 \cdot 2443 = \underbrace{4475576}_{>0}$$

Thus, the sufficient condition for asymptotic stability is fulfilled.



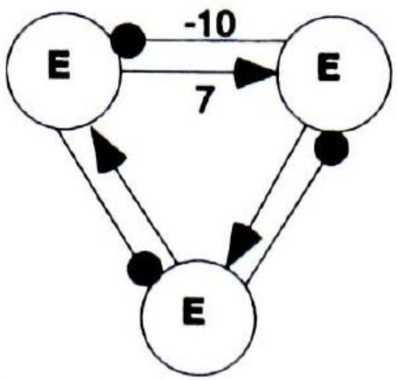
Outline

Theorem 7: how to check for oscillations

Theorem 7 (Routh–Hurwitz criterion for oscillations): For the Nth order system in (4.1) compute the Routh–Hurwitz determinants in Theorem 6. One pair of eigenvalues will be purely imaginary and the system will therefore produce a sinusoidal oscillation if and only if:

$$\Delta_k > 0 \text{ and } \Delta_{N-1} = 0 \text{ for } \quad 1 \leq k \leq N-2$$

Example: checking for oscillations



$$\frac{d}{dt}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} -5 & -10 & 7 \\ 7 & -5 & -10 \\ -10 & 7 & -5 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

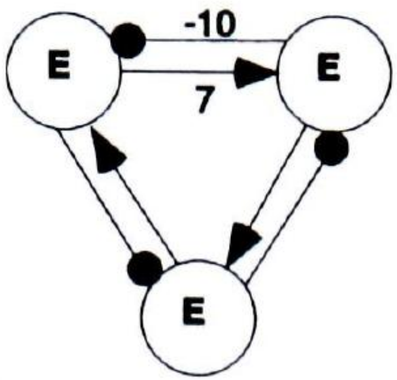
To evaluate whether this system will produce oscillations, we consider two determinants:

$$\Delta_1 = a_1 = \underbrace{15}_{>0}$$

$$\Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = \begin{vmatrix} 15 & 1 \\ 1832 & 285 \end{vmatrix} = \underbrace{2443}_{=0} \boxtimes$$

Thus, this dynamical system will not produce oscillations. (Note that we knew this already from the fact that the system had an asymptotically stable equilibrium point.)

Example: enforcing oscillations (1)



$$\frac{d}{dt}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} -5 & -g & 7 \\ 7 & -5 & -g \\ -g & 7 & -5 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

To find out under what conditions this system will produce oscillations, we require that:

$$\Delta_1 = a_1 > 0$$

$$\Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} = 0$$

The above coefficients are obtained from the characteristic equation:

$$|A - \lambda I| = 0$$

$$\lambda^3 + \underbrace{15}_{a_1}\lambda^2 + \underbrace{(21g + 75)}_{a_2}\lambda + \underbrace{(g^3 + 105g - 218)}_{a_3} = 0$$

Example: enforcing oscillations (2)

Thus, we wish to find g such that:

$$\Delta_1 = 15 > 0$$

$$\Delta_2 = \begin{vmatrix} 15 & 1 \\ g^3 + 105g - 218 & 21g + 75 \end{vmatrix} = 0$$

To satsify the second equation, we require that:

$$15(21g + 75) - (g^3 + 105g - 218) = 0$$
$$g^3 - 210g - 1343 = 0$$

This equation has 1 real solution: $g = 17$.

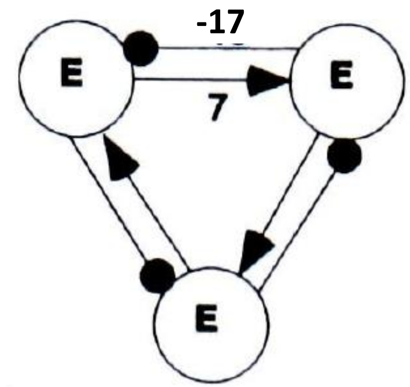
Example: enforcing oscillations (3)

We can use MATLAB to do the above calculations:

```
>> A = '[-5 -g 7; 7 -5 -g; -g 7 -5]';
>> routh_hurwitz(A)

g =
   17.0000

Characteristic_Eqn =
   1.0e+03 *
    0.0010    0.0150    0.4320    6.4800

EigenValues =
        0 +20.7846i
        0 -20.7846i
 -15.0000

RHDeterminants =
   15.0000
         0
    0.0000
ans =

Solution oscillates around equilibrium
point, which is a center.
```

Thus, our system is described by:

$$\frac{d}{dt}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} = \begin{pmatrix}-5 & -17 & 7\\ 7 & -5 & -17\\ -17 & 7 & -5\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix}$$

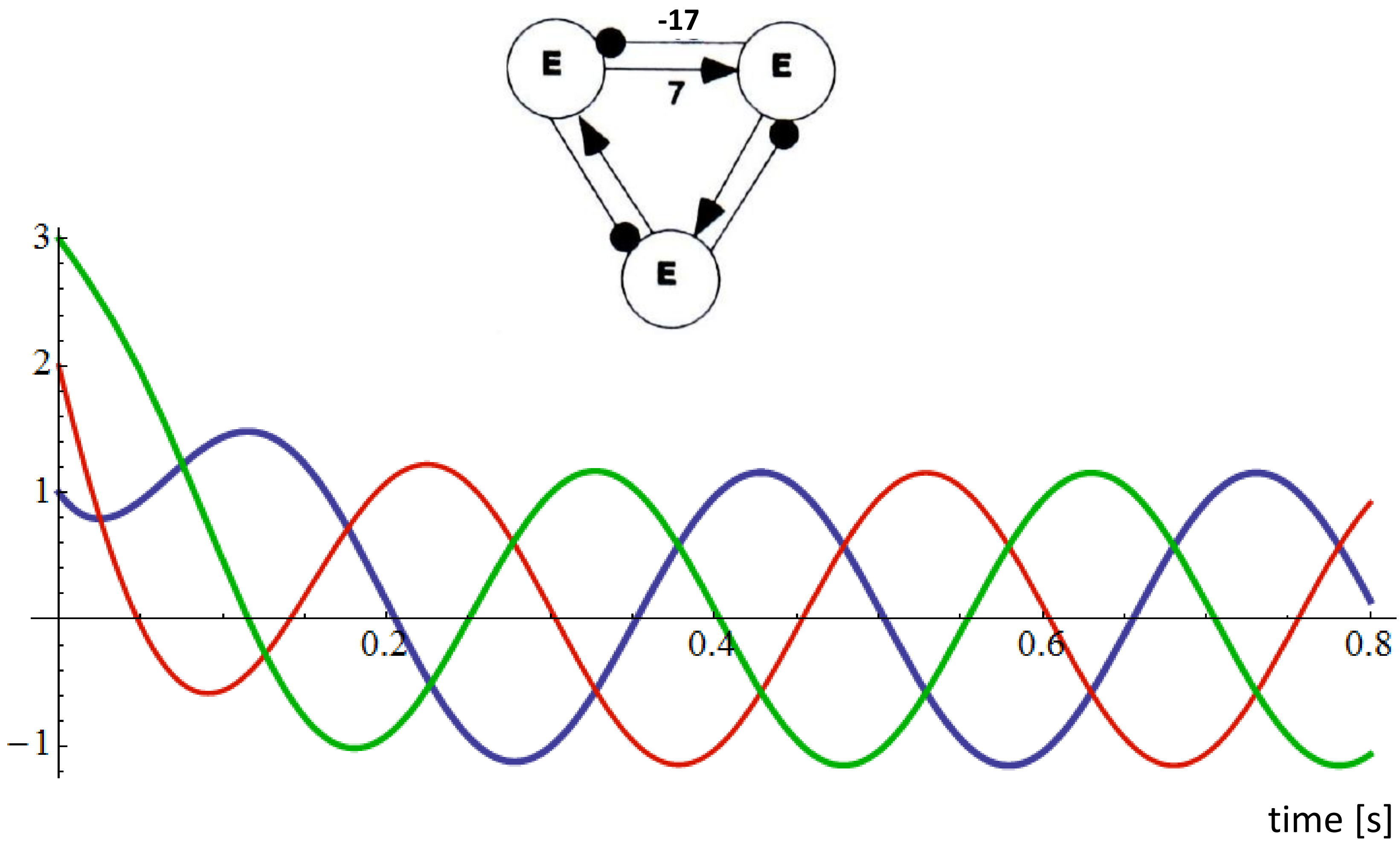


The states have the general form:

$$E_k = Ae^{-15t} + B\cos(20.78t) + C\sin(20.78t)$$

Oscillation frequency: 3.3 Hz

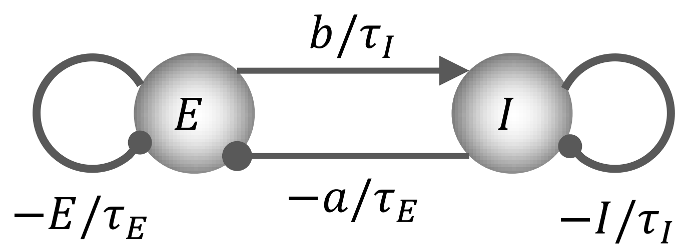
Example: enforcing oscillations (4)



Outline

Oscillations in a two-region network?

Can oscillations even occur in a simpler *two-region* network? We consider a simple negative feedback loop:



The characteristic equation is:

$$\left| \begin{pmatrix} -1/\tau_E & -a/\tau_E \\ b/\tau_I & -1/\tau_I \end{pmatrix} - \lambda I \right| = 0$$

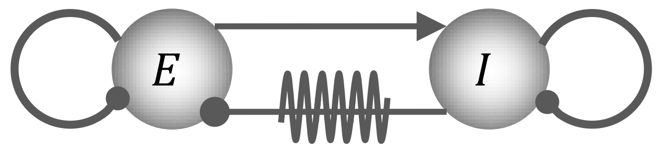
The equation is satisfied by a complex conjugate pair of eigenvalues:

$$\lambda = -\frac{1}{2}\left(\frac{1}{\tau_E} + \frac{1}{\tau_I}\right) \pm \frac{\sqrt{(\tau_E - \tau_I)^2 - 4ab\tau_E\tau_I}}{2\tau_E\tau_I}$$

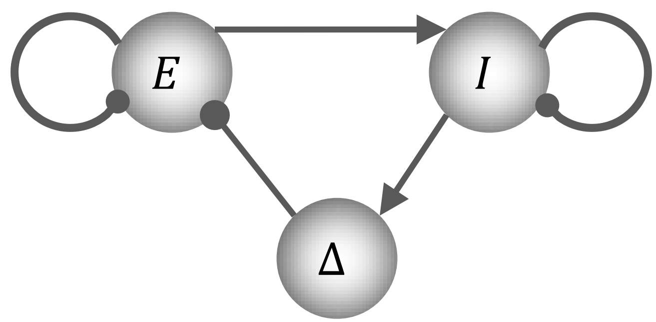
Since the real parts are negative, all solutions must be decaying exponential functions of time, and so oscillations are impossible in this model. But this does not preclude oscillations in second-order systems in general…

Delayed feedback: an approximation

Consider a two-region feedback loop with delayed inhibitory feedback:

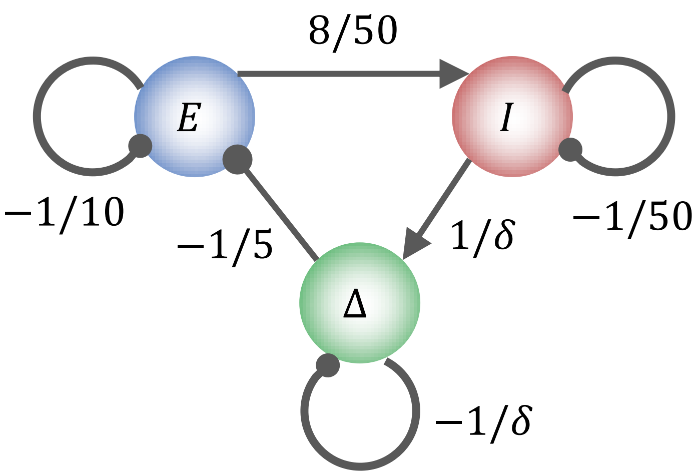


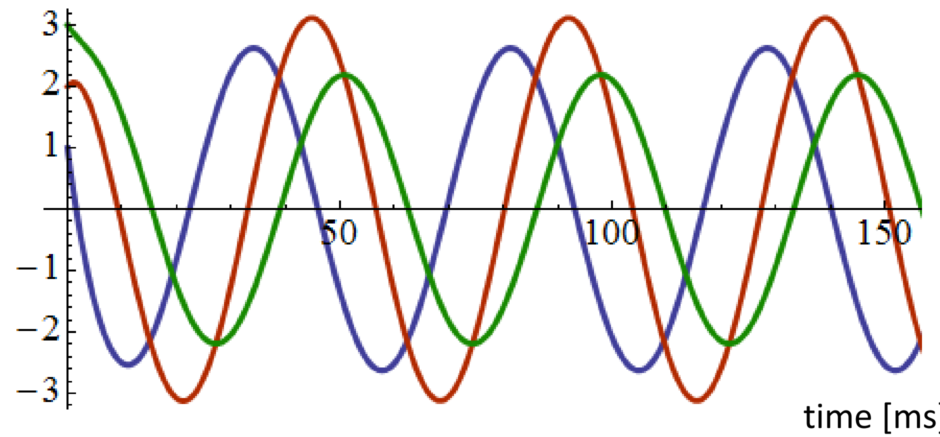
Systems with true delays become exceptionally complex. Instead, we can approximate the effect of delayed feedback by introducing an additional node:



Delayed feedback: enforcing oscillations

Let us specify concrete numbers for all connection strengths, except for the delay:





Is there a value for δ that produces oscillations?

```
>> routh_hurwitz('[-1/10 0 -1/5; 8/50 -1/50 0; 0 1/g -1/g]',10)

g =
    7.6073

Solution oscillates around equilibrium point, which is a center.
```

Summary

Preparations

write down system dynamics in normal form

write down characteristic equation

turn into coefficients form

What are we interested in?

Want to fully solve the system

apply theorem 4 to find eigenvalues λ (may be computationally expensive)

combine with initial conditions to solve for constants A_n

Want to quickly check for asymptotic stability

apply theorem 5 (necessary condition)

- condition not satisfied – not stable
- condition satisfied

apply theorem 6 (sufficient condition)

- condition not satisfied – not stable
- condition satisfied – system is stable

Want to check for, or enforce, oscillations

apply theorem 7 (necessary and sufficient condition)